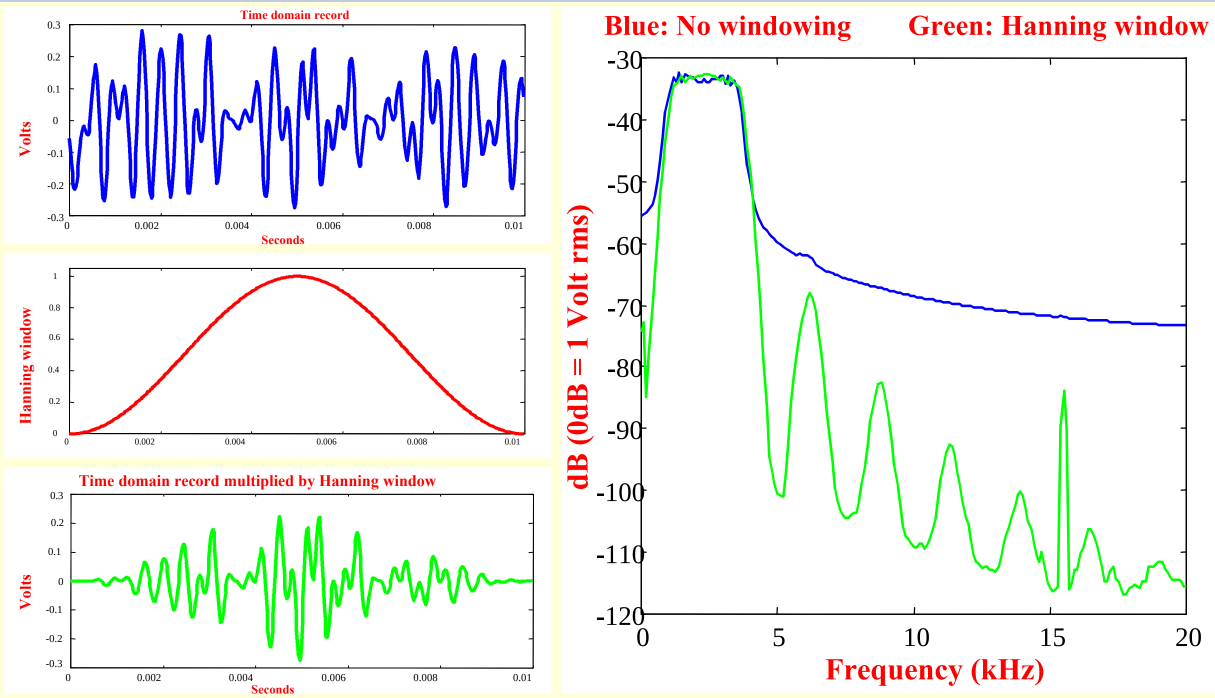
# Windowing: Solves the leakage problem



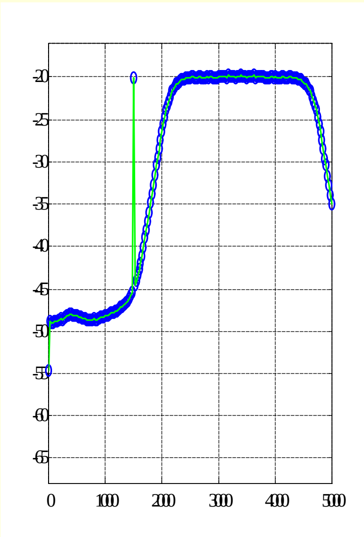

A time history is shown at the upper left. The blue trace on the plot to the right shows the magnitude of the DFT of this time history. The green trace on the same plot is actually a more accurate representation of the signals frequency content. The tone near 16 kHz and the other lumps in the PSD are masked in the DFT by a problem referred to as leakage. The reason is that the blue trace represents the frequency content of the time history repeated an infinite number of times (the periodicity assumptions). Note that the time history begins at a negative voltage but ends at a positive voltage. So when the frame is repeated an placed immediately following, there is a sharp transition from this positive value to the negative one. This sharp transition spreads (or leaks) energy into every bin of the DFT, and obscures the true nature of the frequency content of the signal if we could have sampled it for longer.

Windowing helps solve this problem. Below the time history is the shape of perhaps the most popular window, called Hanning. It is simply a raised cosine function. It is one in the middle and tapers to zero at both ends. If we multiply the time history by this Hanning window, we get the signal in green. Now when we repeat this new signal, no transitions occur at the boundary. The green frequency trace is the magnitude of the DFT of this windowed signal.

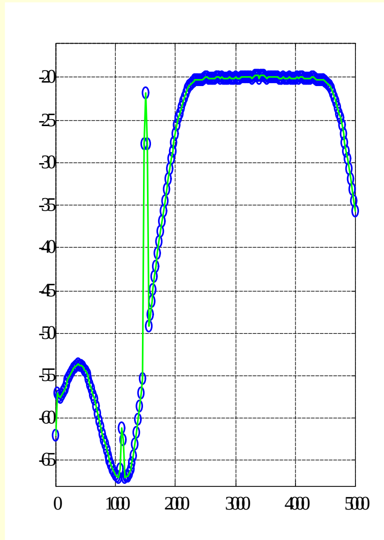
One problem with windowing is that it reduces the total energy in the signal which, as we will see, is difficult to account for.

# Windowing: A blessing and a curse

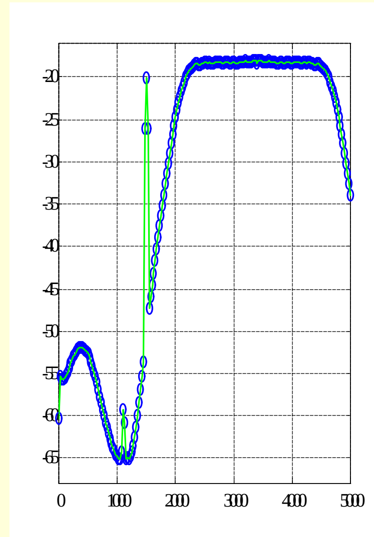
**Boxcar**
**1.5 kHz: -20 dB**
**3 kHz: -20 dB**




**Hanning (Power correction)**
**1.5 kHz: -21.7 dB** ***(1.7dB error)***
**3 kHz: -20 dB**




**Hanning (Amplitude correction)**
**1.5 kHz: -20 dB**
**3 kHz: -18.3 dB** ***(1.7dB error)***




To illustrate a drawback of windowing, I fabricated a signal consisting of a 1.5 kHz sine wave (at -20dBV) superimposed with a narrow band noise source. The PSD is computed using a boxcar window (i.e. no windowing) and displayed on the left. Notice that the sine wave and the noise source show up at the same amplitude (-20dBV).

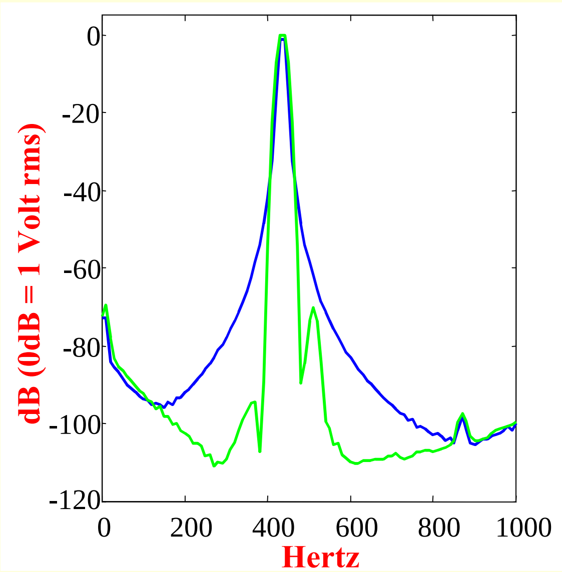
Now we apply the Hanning window (in the frequency domain this can be implemented as a convolution with [-.25 .5 -.25]). The result is shown in the middle figure. Now that the leakage has been reduced by the windowing, we can see that the signal also contains a small 1.1kHz tone. (With the boxcar window, that tone was buried by leakage). The power loss due to this window is 3/8 (sum the squares of kernel), so we have corrected the plot for this power loss by multiplying by 8/3. Notice that the amplitude of the noise source is the same as before. However, when we cursor the peak at the sine wave, we see an amplitude of -21.7 dB. (1.7 dB smaller than before.)

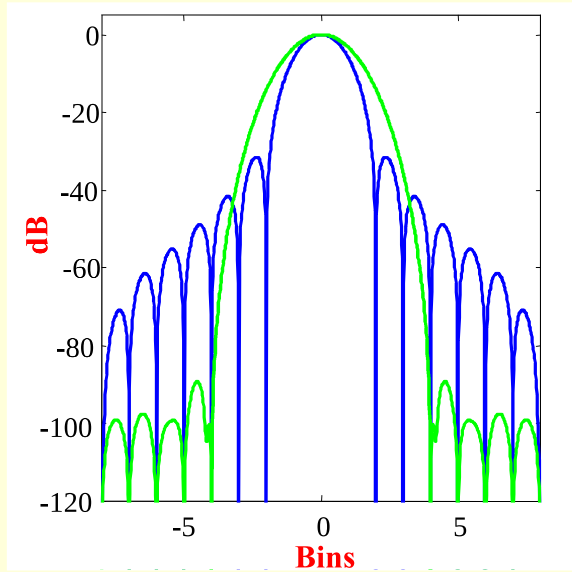
So, what if we wanted to measure the amplitudes of sinusoidal data. Well then we can apply the amplitude correction factor instead (the reciprocal of the square of the center term = 4). With this correction factor we see the display on the right. Now when we cursor the sine wave peak we get the correct amplitude of -20 dBV. But then the amplitude of the noise reads -18.3 dB (again a 1.7 dB error).

Obviously, one correction factor cannot make both numbers right. The windowing has distorted the data in a non-linear way causing the DFT to change shape. (Explain intuitively by showing how Hanning spreads the power of the sine wave out into the neighboring bins.) This same effect is different for the random portion since the neighboring bins are already at roughly the same level.

# Hanning is not the only useful window

**Purple: Hanning     Green: 4-term 92dB Blackman-Harris**




Detecting a signal 70dB down and 7.5 bins away is not possible with Hanning.

Cover right side picture:

This is a PSD of a signal containing 2 sine waves separated by 75 Hz or 7.5 FFT lines ($\Delta f$ = 10Hz).

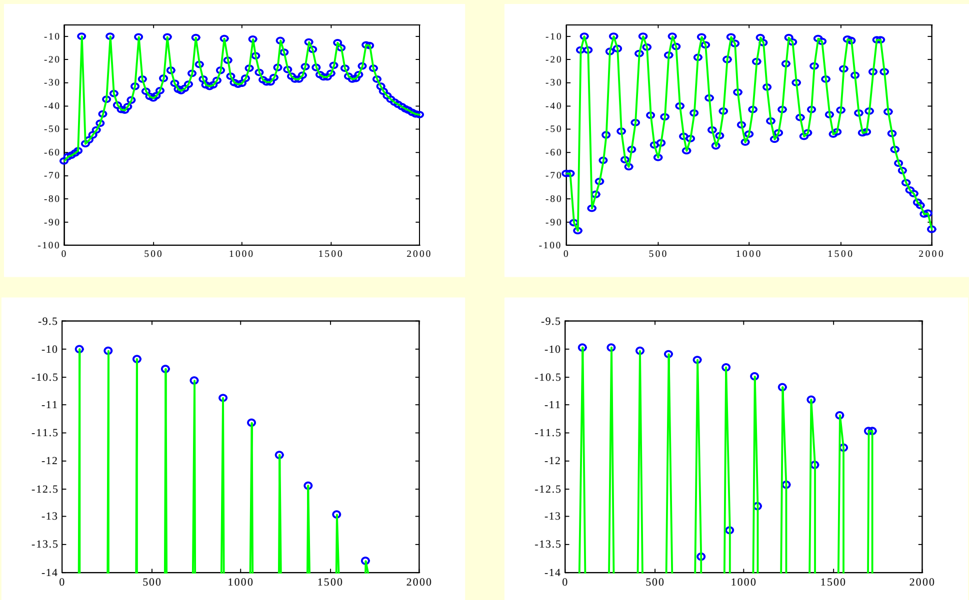
Draw in the shape for the rectangular window. See that the Hanning window provides considerable leakage protection but not enough to detect the smaller sine wave. The Blackman-Harris window easily shows the second sine wave. (Blackman-Harris is actually a family of windows, this one known as the minimum 4-term Blackman-Harris).

Uncover the right picture:

We can see how much leakage protection we get by viewing the Fourier transform of the window shape. The smaller the side lobes, the smaller the leakage. For example, on the Hanning window curve, at 7.5 bins we see that the side lobe is at about -70dB.  But this is the same amplitude as the smaller sine wave, so it is still masked by the leakage. The Blackman-Harris window on the other hand shows about 100 dB of leakage suppression at that position.

The price you pay for the smaller side lobes is the wider main lobe which decreases the frequency resolution of the measurement.

# Hanning also doesn't help much for scalloping loss




Frame=256 points
Δf = 20 Hz

| f (Hz) | f / Δf |
|---|---|
| 100 | 05.00 |
| 261 | 13.05 |
| 422 | 21.10 |
| 583 | 29.15 |
| 744 | 37.20 |
| 905 | 45.25 |
| 1066 | 53.30 |
| 1227 | 61.35 |
| 1388 | 69.40 |
| 1549 | 77.45 |
| 1710 | 85.50 |

Boxcar: Scallop loss = 3.9 dB
Highest side lobe = 13 dB

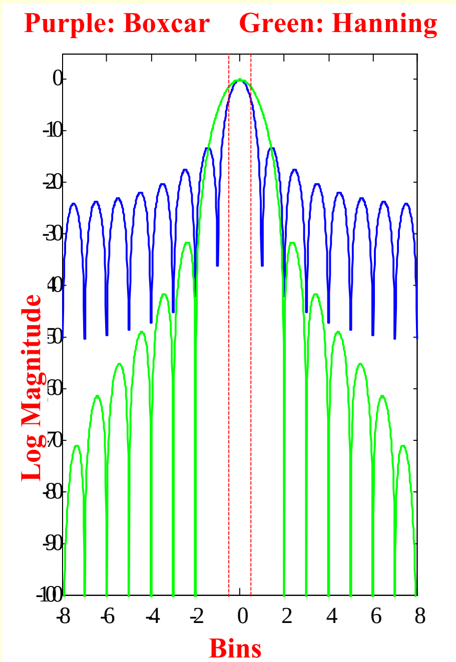
Hanning: Scallop loss = 1.4 dB
Highest side lobe = 32 dB

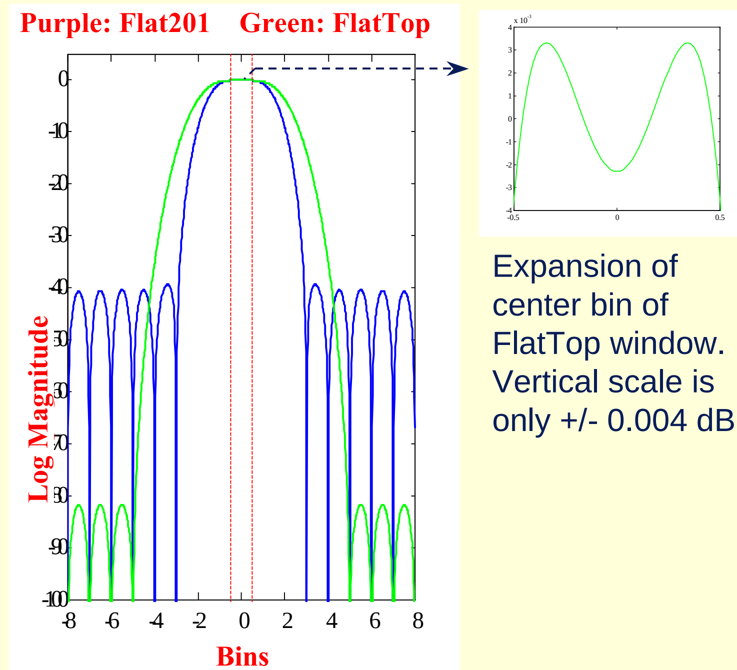
Another problem with windowing (and of FFT analysis in general) is scalloping loss. In my struggle to demonstrate this phenomenon, I fabricated a signal consisting of the sum of 11 equal amplitude sine waves at the frequencies listed here (from 100 Hz up to 1710 Hz). If we sample 256 points of this signal at a 5120 Hz sample rate, our FFT line spacing is Δf = 20 Hz. So dividing by Δf we get the bin numbers shown here. The important part is the fractional component shown in blue. Note that the first component, 100 Hz is exactly on a line (bin 5) and the last component, 1710 is exactly between two lines (bin 85.5), and all the components in between slowly shift from these two extremes as can be seen by looking at the fractional part of the bin number.

Now the PSD (no windowing) is displayed in the upper left. The high leakage is especially noticeable near 2 kHz. Otherwise the display looks like a good representation of the signal, with one peak for each sine wave component. However if we expand the amplitude scale (near -10 dB) we see that the amplitudes are not the same for all components as you might expect. The first term (100Hz) is actually correct. The others are in error (up to 3.9 dB) due to scalloping loss.]

When we repeat the experiment with the Hanning window, we see that the leakage is much reduced, but the scalloping loss is still significant (a bit less at 1.4 dB).

# FlatTop windows to the rescue




Expansion of center bin of FlatTop window. Vertical scale is only +/- 0.004 dB

Cover right side:

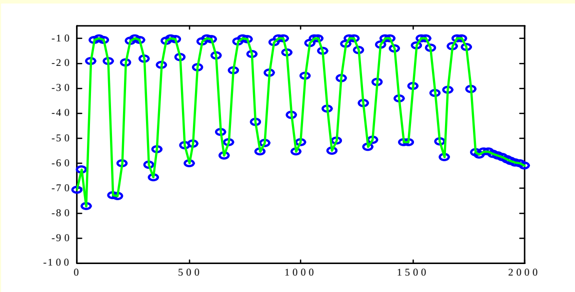
Why do we get this scalloping loss? The answer is easily seen from the Fourier transform of the window shapes (shown here for rectangular and Hanning). The red dotted lines show the extend of the center bin. The amount of droop within these dotted lines represents the scalloping loss. You can see that Hanning shows less droop than the boxcar.

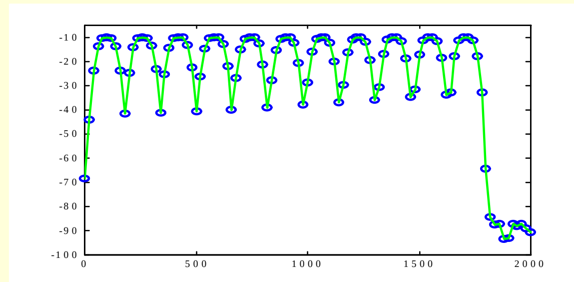
Uncover right side:

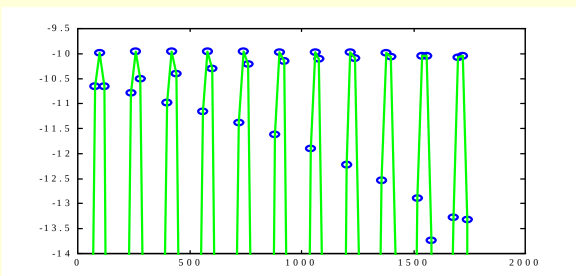
Now lets look at the transform of two windows that do not have appreciable scalloping loss. The first one (blue) I'll call the "Flat201" window (stolen from the Potter 201) and the second one (green) I'll just call "FlatTop". (Note: FlatTop refers to a generic window style, and not to specific window coefficients. So one instrument's FlatTop may be different than the next.)

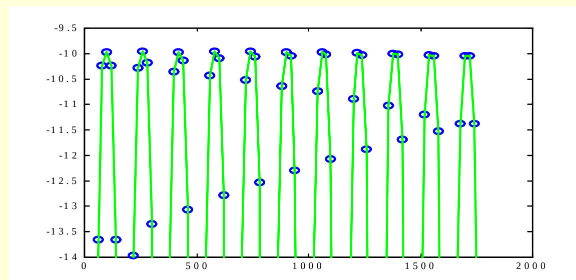
The main difference between these two windows is in how they play the tradeoff between leakage suppression and the width of the main lobe. Both reduce scalloping loss to negligible levels (.01 and .0035 dB respectively). Here is an expanded view of the center bin for the FlatTop window (Flat201 looks similar). Not that the vertical scale used here is very small (+/- .004 dB).

# FlatTop windows: The proof




Flat201: Scallop loss = ±0.01 dB
Highest side lobe = 40 dB

FlatTop: Scallop loss = ±0.0035 dB
Highest side lobe = 82 dB

Now here is the proof.

We again compute the PSD of the signal containing the 11 sine waves, but now using these two FlatTop windows. The lower plots show the same expanded view as before, and there is no scalloping loss visible.

We can see that the leakage suppression on the right side (FlatTop) is much better than on the left (Flat201). But note how the width of the main lobe affects the result. In the display using the FlatTop, the 11 peaks are more smeared together due to the wider main lobe.